```
--> kill(all)$
    load(lrats)$
```

Central Force of motion (=diff eq),version 1.0a
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rem : find F(r) ?, orbit is known.
----

from Lagrangain equation : r-coordinate.
=======================================

```
--> p1:μ·('diff(r,t,2)−r·('diff(θ,t))^2) =
    F(r);
```

(p1) $$\left(\frac{d^2}{d t^2} r - r\left(\frac{d}{d t}\theta\right)^2\right)\mu = F(r)$$

build u : diff eq
============== .

```
--> p2:u=1/r;
```

(p2) $$u = \frac{1}{r}$$

part 1:du/dθ : ?
----

u(θ) ⇒ u(r)*r(θ).r(t).θ(t)

1e)

du/dθ =du/dr*dr/dθ = du/dr*dr/dθ = du/dr*dr/dt*dt/dθ.

```
--> p3:'diff(u,θ) =
    'diff(u,r)·'diff(r,t)/'diff(θ,t);
```

(p3) $$\frac{d}{d\theta} u = \frac{\left(\frac{d}{d t} r\right)\left(\frac{d}{d r} u\right)}{\frac{d}{d t}\theta}$$

```
--> p3a: 'diff(u,r) =
    subst([p2],'diff(u,r));
```

(p3a) $$\frac{d}{d r} u = \frac{d}{d r}\frac{1}{r}$$

```
--> p3b:lhs(p3a) =
```

ev(rhs(p3a),diff);

(p3b) $$\frac{d}{d\,r}u=-\frac{1}{r^2}$$

--> p3c:subst([p3b],p3);

(p3c) $$\frac{d}{d\,\theta}u=-\frac{\frac{d}{d\,t}r}{r^2\left(\frac{d}{d\,t}\theta\right)}$$

2e)

use definition : l
----

find : dθ/dt

--> p4:l = μ·r^2·'diff(θ,t);

(p4) $$l=r^2\left(\frac{d}{d\,t}\theta\right)\mu$$

--> p4a:solve(p4,
('diff(θ,t,1)));

(p4a) $$[\frac{d}{d\,t}\theta=\frac{l}{r^2\,\mu}]$$

insert 'p4a' into 'p3c'

--> p4b:lhs(p3c) = subst(p4a,rhs(p3c));

(p4b) $$\frac{d}{d\,\theta}u=-\frac{\left(\frac{d}{d\,t}r\right)\mu}{l}$$

part 2:diff(θ,u,2) : ?

--> p5:'diff(u,θ,2) =
'diff(rhs(p4b),θ);

(p5) $$\frac{d^2}{d\,\theta^2}u=\frac{d}{d\,\theta}\left(-\frac{\left(\frac{d}{d\,t}r\right)\mu}{l}\right)$$

rem : d/dθ = d/dt*dt/dθ = d/dt (1/dθ/dt)
===================================

--> p5a:lhs(p5) = 1/'diff(θ,t)·'diff(rhs(p4b),t);

(p5a) $$\frac{d^2}{d\,\theta^2}u=\frac{\frac{d}{d\,t}\left(-\frac{\left(\frac{d}{d\,t}r\right)\mu}{l}\right)}{\frac{d}{d\,t}\theta}$$

use definition : l , use 'p4a'

--> p5b:lhs(p5a)=subst(p4a,rhs(p5a));

(p5b) $$\frac{d^2}{d\,\theta^2}u=\frac{r^2\mu\left(\frac{d}{d\,t}\left(-\frac{\left(\frac{d}{d\,t}r\right)\mu}{l}\right)\right)}{l}$$

ev : diff

--> depends(r,t);

(%o14) [r(t)]

depends must be used otherwise : rhs(p5b) = 0

--> p5c:lhs(p5b)=ev(rhs(p5b),diff) ;

(p5c) $$\frac{d^2}{d\,\theta^2}u=-\frac{r^2\left(\frac{d^2}{d\,t^2}r\right)\mu^2}{l^2}$$

--> p5d:solve(p5c, ('diff(r,t,2)));

(p5d) $$\left[\frac{d^2}{d\,t^2}r=-\frac{l^2\left(\frac{d^2}{d\,\theta^2}u\right)}{r^2\mu^2}\right]$$

part 3: ' use above ' fill in 'p1' :?

1e) fill in:
=======

```
--> p6:subst(p4a,p1)$

--> p6a:subst(p5d,p6);
```

(p6a) $$\left(-\frac{l^2\left(\frac{d^2}{d\,\theta^2}u\right)}{r^2\,\mu^2}-\frac{l^2}{r^3\,\mu^2}\right)\mu=F(r)$$

```
--> p6b:expand(p6a);
```

(p6b) $$-\frac{l^2\left(\frac{d^2}{d\,\theta^2}u\right)}{r^2\,\mu}-\frac{l^2}{r^3\,\mu}=F(r)$$

2e) find coef : 'diff(u,θ,2), lhs
========================

```
--> p6c:ratcoeff(lhs(p6b),'diff(u,θ,2)
    );
```

(p6c) $$-\frac{l^2}{r^2\,\mu}$$

```
--> p6c1:lratsubst([rhs(p2)=lhs(p2)],p6c)
    ;
```

(p6c1) $$-\frac{l^2\,u^2}{\mu}$$

3e) find coef : without 'diff(u,θ,2),lhs
==============================

```
--> p6d:ev(lhs(p6b),
    ('diff(u,θ,2))=0);
```

(p6d) $$-\frac{l^2}{r^3\,\mu}$$

```
--> p6d1:lratsubst([rhs(p2)=lhs(p2)],p6d)
    ;
```

(p6d1) $$-\frac{l^2\,u^3}{\mu}$$

4e) into lhs(p1)
===============

--> p6e:p6c1·'diff(u,θ,2)+p6d1 = rhs(p1);

(p6e) $$-\frac{l^2 u^2 \left(\frac{d^2}{d\,\theta^2} u\right)}{\mu} - \frac{l^2 u^3}{\mu} = F(r)$$

--> p6f:1/p6c1·p6e;

(p6f) $$-\frac{\left(-\frac{l^2 u^2 \left(\frac{d^2}{d\,\theta^2} u\right)}{\mu} - \frac{l^2 u^3}{\mu}\right)\mu}{l^2 u^2} = -\frac{F(r)\,\mu}{l^2 u^2}$$

--> p6g:ratsimp(p6f);

(p6g) $$\frac{d^2}{d\,\theta^2} u + u = -\frac{F(r)\,\mu}{l^2 u^2}$$

part 4: solve 'p6g'

1e) F(r) = -div(V) = -k/r^2
=====================
V = potential = V(r), divergence = div

--> p7:subst([F(r)=−k·u^2],p6g);

(p7) $$\frac{d^2}{d\,\theta^2} u + u = \frac{k\,\mu}{l^2}$$

--> p7a:ratsimp(p7);

(p7a) $$\frac{d^2}{d\,\theta^2} u + u = \frac{k\,\mu}{l^2}$$

2e) ode2, u
==========

1e) u = u(θ)
=========

--> p7b:ode2(p7a,u,θ);

(p7b) $$u = \frac{k\,\mu}{l^2} + \%k1 \sin(\theta) + \%k2 \cos(\theta)$$

--> p7c:subst([p2],p7b);

(p7c) $$\frac{1}{r} = \frac{k\,\mu}{l^2} + \%k1 \sin(\theta) + \%k2 \cos(\theta)$$

rem : %k1*sin(θ)+%k2*cos(θ) = d*cos(θ-θo);
d,θo = find out ?

take axis so that : θo =0

--> p7b:lhs(p7c)=rhs(p7a)+d·cos(θ);

(p7b) $$\frac{1}{r}=\frac{k\,\mu}{l^2}+d\cos(\theta)$$

--> p7c:solve(p7b,r)[1];

(p7c) $$r=\frac{l^2}{k\,\mu+d\,l^2\cos(\theta)}$$

--> p7c1:part(rhs(p7c),1);
p7c2:part(rhs(p7c),2);

(p7c1) l^2 (p7c2) $k\,\mu+d\,l^2\cos(\theta)$

--> p7d1: p7c1/(k·μ);
p7d2:p7c2/(k·μ);

(p7d1) $$\frac{l^2}{k\,\mu}$$ (p7d2) $$\frac{k\,\mu+d\,l^2\cos(\theta)}{k\,\mu}$$

--> p7e1:(p7d1);
p7e2:expand(p7d2);

(p7e1) $$\frac{l^2}{k\,\mu}$$ (p7e2) $$\frac{d\,l^2\cos(\theta)}{k\,\mu}+1$$

--> p7f:lhs(p7c)= p7e1/p7e2;

(p7f) $$r=\frac{l^2}{k\left(\frac{d\,l^2\cos(\theta)}{k\,\mu}+1\right)\mu}$$

p = semi-latus rectum = 'p7e1'

--> p7g:lhs(p7f) = p/p7e2;

(p7g) $$r=\frac{p}{\frac{d\,l^2\cos(\theta)}{k\,\mu}+1}$$

ε = (d*l^2)/(k*μ)

--> p7h:ε = (d·l^2)/(k·μ);

(p7h) $$\varepsilon=\frac{d\,l^2}{k\,\mu}$$

--> p7i:
lratsubst([rhs(p7h)=lhs(p7h)],p7g);

(p7i) $$r=\frac{p}{\varepsilon\cos(\theta)+1}$$

rem : polar equation of conic section(ellipse,...) = 'p7i'

rem : find d ? uniquely determines , given E, angular momentum.

We can use : 'p6g' when r=r(θ), known orbit.
use : u = 1/r
⇒ Find force law.

Example : r= r(θ)
⇒ r=k*exp(θ)
Find : F(r)

start :
=====

--> p8:p6g;

(p8) $$\frac{d^2}{d\,\theta^2}u+u=-\frac{F(r)\,\mu}{l^2\,u^2}$$

--> p8a:subst([p2],p8);

(p8a) $$\frac{1}{r}+\frac{d^2}{d\,\theta^2}\frac{1}{r}=-\frac{r^2\,F(r)\,\mu}{l^2}$$

use : equation of orbit. (into p8a) :
==============================

--> p8b:subst([r=k·exp(θ)],p8a);

(p8b) $$\frac{d^2}{d\,\theta^2}\frac{\%e^{-\theta}}{k}+\frac{\%e^{-\theta}}{k}=-\frac{k^2\,\%e^{2\,\theta}\,F\left(k\,\%e^{\theta}\right)\mu}{l^2}$$

--> p8b1:k·exp(θ) = r;

(p8b1) $$k\,\%e^{\theta}=r$$

--> p8c:ev(p8b,diff);

(p8c) $$\frac{2\,\%e^{-\theta}}{k}=-\frac{k^2\,\%e^{2\,\theta}\,F\left(k\,\%e^{\theta}\right)\mu}{l^2}$$

--> p8d:solve(p8c,F(k·%e^θ))[1];

(p8d) $$F\left(k\,\%e^{\theta}\right)=-\frac{2\,l^2\,\%e^{-3\,\theta}}{k^3\,\mu}$$

Force Law : F(r)
=============

--> p8e:subst([p8b1],lhs(p8d)) = rhs(p8d);

(p8e) $$F(r)=-\frac{2\,l^2\,\%e^{-3\,\theta}}{k^3\,\mu}$$

rem : removes θ,from rhs 'p8e'

--> p8f:solve(p8b1,θ);

(p8f) $[\theta = \log\left(\frac{r}{k}\right)]$

use ' p8f ' into 'p8e'
================

--> p8g:subst(p8f,p8e);

(p8g) $F(r) = -\frac{2\,l^2}{r^3\,\mu}$

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