maxima: orbit equation , version 1.0
peter.vlasschaert@gmail.com,18/12/2023

load packages,option

```
load(lrats)$
ratprint:false$
```

General equation : (orbit or trajectory) of space vehicle
==========

part 1 : Energy
---

step 1:kinetic energy:T
---

from mechanics : T=1/2*m*V^2, kinetic energy from motion.

```
p1a: T=1/2·m·V^2;
```

$$T=\frac{V^2\,m}{2}$$

m: small mass
V: velocity of small body around the large mass

a) polair equation for V, velocity V [m/s]

```
p1a1:V^2=('diff(r,t))^2+(r·'diff(θ,t))^2;
```

$$V^2=r^2\left(\frac{d}{d\,t}\,\theta\right)^2+\left(\frac{d}{d\,t}\,r\right)^2$$

r: distance between large mass and small mass

b) final kinetic : equation

```
p1a2:subst([p1a1],p1a);
```

$$T=\frac{m\left(r^2\left(\frac{d}{d\,t}\,\theta\right)^2+\left(\frac{d}{d\,t}\,r\right)^2\right)}{2}$$

```
p1a21:expand(p1a2);
```

$$T=\frac{m\,r^2\left(\frac{d}{d\,t}\,\theta\right)^2}{2}+\frac{m\left(\frac{d}{d\,t}\,r\right)^2}{2}$$

step 2: pontential energy:P
---

from mechanics : P

a) (F)orce : motion small mass:m around gravitation field of large mass :M

```
p1b:F=G·(M·m)/r^2; [N : kg·m/s^2];
```

$$F=\frac{G\,M\,m}{r^2}\qquad\left[\frac{kg\,m}{s^2}\right]$$

G: universal gravitational constant:6.67*10^(-11) m^3/(kg)*(s)^2
M: mass large mass , earth → 5.98*10^24 kg
m: small mass

```
p1ba1:Γ=G·M;
```

$$\Gamma=G\,M$$

```
p1ba2:reverse(p1ba1);
```

$$G\,M=\Gamma$$

b) P : integration of (F)orce :from infinity to r

```
assume(r>0);
```

$$[r>0]$$

```
p1b1:''integrate(1,P,0,P)=-integrate(rhs(p1b),r,∞,r);
```

$$P=-\left(G\,M\,m\int_r^{\infty}\frac{1}{r^2}\,dr\right)$$

```
p1b1a:lratsubst([reverse(p1ba1)],ev(p1b1,nouns));
```

$$P=-\left(\frac{m\,\Gamma}{r}\right)$$

step 3: E ,total Energy

```
p1c:E= T+P;
```

$$E=T+P$$

use : p1a2 & p1b1a

a) Energy :E [Joule =J=N*m]

```
p1c1:subst([p1a2,p1b1a],p1c);
```

$$E=\frac{m\left(r^2\left(\frac{d}{dt}\theta\right)^2+\left(\frac{d}{dt}r\right)^2\right)}{2}-\frac{m\,\Gamma}{r}$$

b) Energy per unit mass :Em=E/m

```
p1c2:Em =expand( rhs(p1c1)/m);
```

$$Em=\frac{r^2\left(\frac{d}{dt}\theta\right)^2}{2}-\frac{\Gamma}{r}+\frac{\left(\frac{d}{dt}r\right)^2}{2}$$

part 2: (La)grangian & (La)grangian (eq)uation

→ (La)grangian

```
p2a:La= T−P;
p2a1:La=subst([p1a2,p1b1a],rhs(p2a));
```

$$La=T-P \qquad La=\frac{m\left(r^2\left(\frac{d}{dt}\theta\right)^2+\left(\frac{d}{dt}r\right)^2\right)}{2}+\frac{m\,\Gamma}{r}$$

→ (La)grangian (eq)uation

```
depends([θ,r],t);
```

$$[\theta(t),r(t)]$$

```
p2b:Laeq(va,B):=diff(diff(B,diff(va,t)),t)−diff(B,va) =0;
```

$$\mathrm{Laeq}(va,B):=\frac{d}{dt}\left(\frac{d}{d\left(\frac{d}{dt}va\right)}B\right)-\frac{d}{d\,va}B=0$$

→ va:θ, Laeq(La,va):

```
p2b1a:Laeq(θ,rhs(p2a1));
```

$$m\,r^2\left(\frac{d^2}{dt^2}\theta\right)+2\,m\,r\left(\frac{d}{dt}r\right)\left(\frac{d}{dt}\theta\right)=0$$

```
p2b2a:'diff((m·r^2·'diff(θ,t)),t) =diff((m·r^2·'diff(θ,t)),t);
```

$$\frac{d}{dt}\left(m\,r^2\left(\frac{d}{dt}\theta\right)\right)=m\,r^2\left(\frac{d^2}{dt^2}\theta\right)+2\,m\,r\left(\frac{d}{dt}r\right)\left(\frac{d}{dt}\theta\right)$$

final result : va = θ

```
p2b2b:lratsubst([p2b1a],p2b2a);
```

$$\frac{d}{dt}\left(m\,r^2\left(\frac{d}{dt}\theta\right)\right)=0$$

rule : d(A)/dt = 0, A = constant => A=angular momentum.

```
p2b2c:part(lhs(p2b2b),1) = A;
```

$$m\,r^2\left(\frac{d}{dt}\theta\right)=A$$

→ va:r, Laeq(La,va):

```
p2c1:Laeq(r,rhs(p2a1));
```

$$-\left(m\,r\left(\frac{d}{dt}\theta\right)^2\right)+\frac{m\,\Gamma}{r^2}+m\left(\frac{d^2}{dt^2}r\right)=0$$

definition :angularmomentum per unit mass

```
p2c2:h=r^2·diff(θ,t);
```

$$h=r^2\left(\frac{d}{dt}\theta\right)$$

→ use :p2c1*r^3/m

```
p2c3:expand(p2c1·r^3/m);
```

$$-\left(r^4\left(\frac{d}{dt}\theta\right)^2\right)+r\,\Gamma+r^3\left(\frac{d^2}{dt^2}r\right)=0$$

→ use : p2c2 into p2c3 => */r^3

```
p2c4:expand(lratsubst([reverse(p2c2)],p2c3)/r^3);
```

$$\frac{\Gamma}{r^2}+\frac{d^2}{dt^2}r-\frac{h^2}{r^3}=0$$

r=1/u , h=diff(θ,t)/u^2

```
depends([u],t);
```

$$[u(t)]$$

< dr/dt = ?> , first order diferential from r with respect to t

→ dr/dt = ? ,subst r= 1/u

```
p4d1:subst([r=1/u],diff(r,t,1));
```

$$\frac{d}{d\,t}\frac{1}{u}$$

```
p4d1:ev(p4d1,nouns);
```

$$-\left(\frac{\frac{d}{d\,t}u}{u^2}\right)$$

```
depends([u],θ);
```

$$[u(\theta)]$$

→du/dt = ? ,subst du/dt = du/dθ*dθ/dt

```
p4d2:lratsubst([diff(u,t)=diff(u,θ)·diff(θ,t)],ev(p4d1,nouns));
```

$$-\left(\frac{\left(\frac{d}{d\,\theta}u\right)\left(\frac{d}{d\,t}\theta\right)}{u^2}\right)$$

→ use h , p2c2

```
p4d3:subst([r=1/u],p2c2);
```

$$h=\frac{\frac{d}{d\,t}\theta}{u^2}$$

→ subst p4d3 into p4d2

```
p4d4:lratsubst([reverse(p4d3)],p4d2);
```

$$-\left(h\left(\frac{d}{d\,\theta}u\right)\right)$$

< d(dr/dt)/dt = ?> , second order diferential from r with respect to t

use :p4d4

```
p4e1:diff(r,t,2)=diff(p4d4,t);
```

$$\frac{d^2}{d\,t^2}r=-\left(h\left(\frac{d^2}{d\,\theta^2}u\right)\left(\frac{d}{d\,t}\theta\right)\right)$$

use : p4d3 ,solve : diff(θ,t)

```
p4e1a:solve(p4d3,diff(θ,t));
```

$$\left[\frac{d}{d\,t}\theta=h\,u^2\right]$$

```
p4e2:subst(%,p4e1);
```

$$\frac{d^2}{d\,t^2}r=-\left(h^2\,u^2\left(\frac{d^2}{d\,\theta^2}u\right)\right)$$

part 3: final differential equation computing orbit equation

---

use :p2c4

```
p5f1:p2c4;
```

$$\frac{\Gamma}{r^2}+\frac{d^2}{d\,t^2}r-\frac{h^2}{r^3}=0$$

use : p4e2 into p2c4

```
p5f2:subst([p4e2],p5f1);
```

$$\frac{\Gamma}{r^2}-h^2\,u^2\left(\frac{d^2}{d\,\theta^2}u\right)-\frac{h^2}{r^3}=0$$

```
p5f2a:coeff(lhs(p5f2),diff(u,θ,2));
```

$$-(h^2\,u^2)$$

use :p5f2/p5f2a

```
p5f3:expand(p5f2/p5f2a);
```

$$-\left(\frac{\Gamma}{h^2\,r^2\,u^2}\right)+\frac{d^2}{d\,\theta^2}u+\frac{1}{r^3\,u^2}=0$$

use: p5f3 , subst r=1/u

```
p5f4:subst([r=1/u],p5f3);
```

$$-\left(\frac{\Gamma}{h^2}\right)+\frac{d^2}{d\,\theta^2}u+u=0$$

```
p5f5:ode2(p5f4,u,θ);
```

$$u=\%k1\,\sin(\theta)+\%k2\,\cos(\theta)+\frac{\Gamma}{h^2}$$

part 4:find polair equation , u =1/r.

---

form : r=p/(1+e*cos(θ-C))

subst: u=1/r into p5f5

```
p6e1:subst([u=1/r],p5f5);
```

$$\frac{1}{r}=\%k1\sin(\theta)+\%k2\cos(\theta)+\frac{\Gamma}{h^2}$$

```
p6e1a:r=p/(1+e·cos(θ−C));/*ref for conic section*/
```

$$r=\frac{p}{e\cos(\theta-C)+1}$$

e:eccentricity

==========
if e=0 the orbit is a circle
if e<1 the orbit is an ellipse
if e=1 the orbit is a parabola
if e> 1 the orbit is a hyperbola
==========
C = phase angle ,below pos x-axis, "ref : angles "
θ = true anomaly,
r=p ,as θ-BB =π/2
==========

solve p6e1 for r

```
p6e2:solve(p6e1,r);
```

$$\left[r=\frac{h^2}{\%k1\,h^2\sin(\theta)+\%k2\,h^2\cos(\theta)+\Gamma}\right]$$

differential eq : second order two constant we must find

use :numerator of r

```
p6e2a:num(rhs(p6e2[1]))/Γ;/*find form :p6e1a*/
```

$$\frac{h^2}{\Gamma}$$

use : denominator of r

```
p6e2aa:expand(denom(rhs(p6e2[1])));
```

$$\%k1\,h^2\sin(\theta)+\%k2\,h^2\cos(\theta)+\Gamma$$

use : {%k1,%k2} →{AA,BB}

```
p6e2a1:p6e2aa =AA·h^2·cos(θ−BB)+Γ;/*find form :p6e1a*/
```

$$\%k1\,h^2\sin(\theta)+\%k2\,h^2\cos(\theta)+\Gamma=AA\,h^2\cos(\theta-BB)+\Gamma$$

```
p6e2a2:expand(p6e2a1/Γ);/*find form :p6e1a*/
```

$$\frac{\%k1\,h^2\sin(\theta)}{\Gamma}+\frac{\%k2\,h^2\cos(\theta)}{\Gamma}+1=\frac{AA\,h^2\cos(\theta-BB)}{\Gamma}+1$$

use : p6e2a/p6e2a2

```
p6e3: r =p6e2a/rhs(p6e2a2);
```

$$r=\frac{h^2}{\Gamma\left(\frac{AA\,h^2\cos(\theta-BB)}{\Gamma}+1\right)}$$

1e) r=p , when true anomaly , θ-BB =π/2

```
p6e3a: subst([r=p, θ−BB =π/2],p6e3);
```

$$p=\frac{h^2}{\Gamma}$$

```
pe3a1:lratsubst([reverse(p6e3a)],p6e3);
```

$$r=\frac{p}{AA\,p\cos(\theta-BB)+1}$$

see:p6e1a:r=p/(1+e*cos(θ-C));/*ref for conic section,see my website*/

2e) AA.p =e

```
pe3b:lratsubst([AA·p =e],pe3a1);
```

$$r=\frac{p}{e\cos(\theta-BB)+1}$$

rem :

```
pe3c:(denom(rhs(pe3b))·r=p)^2;
```

$$r^2\left(e\cos(\theta-BB)+1\right)^2=p^2$$

```
pe3c1:reverse(pe3c);
```

$$p^2=r^2\left(e\cos(\theta-BB)+1\right)^2$$

part 5: How to find e , for orbit use E or Em
---
use : E = T+P , p1c ,conservation energy

a) T use "conic section : r"

use : compute velocity V

```
pd1:diff(pe3b,t);
```

$$\frac{d}{dt}r=\frac{e\,p\left(\frac{d}{dt}\theta\right)\sin(\theta-BB)}{(e\cos(\theta-BB)+1)^2}$$

```
pd1a: part(rhs(p1c),1)=(subst([pd1],rhs(p1a21)));
```

$$T=\frac{e^2\,m\,p^2\left(\frac{d}{dt}\theta\right)^2\sin(\theta-BB)^2}{2\,(e\cos(\theta-BB)+1)^4}+\frac{m\,r^2\left(\frac{d}{dt}\theta\right)^2}{2}$$

use :pe3c1

```
pd1b:subst([pe3c1],pd1a);
```

$$T=\frac{e^2\,m\,r^2\left(\frac{d}{dt}\theta\right)^2\sin(\theta-BB)^2}{2\,(e\cos(\theta-BB)+1)^2}+\frac{m\,r^2\left(\frac{d}{dt}\theta\right)^2}{2}$$

use :p2c2

```
pdd: subst(solve(p2c2,diff(θ,t)),pd1b);
```

$$T=\frac{e^2\,h^2\,m\sin(\theta-BB)^2}{2\,r^2\,(e\cos(\theta-BB)+1)^2}+\frac{h^2\,m}{2\,r^2}$$

use : pe3c

```
pd1c:factor(trigsimp(factor(lratsubst([pe3c],pdd))));
```

$$T=\frac{h^2\,m\,(2\,e\cos(\theta-BB)+e^2+1)}{2\,p^2}$$

combination : factor(trigsimp(factor(?)))
use : trig function are inside

b) P use "conic section : r"

use :p1b1a

```
pd1d:p1b1a;
```

$$P=-\left(\frac{m\,\Gamma}{r}\right)$$

use :p6e1a

```
pd1da:subst([p6e1a],pd1d);
```

$$P=-\left(\frac{m\,\Gamma\,(e\cos(\theta-C)+1)}{p}\right)$$

use:p1c1

```
pdf:subst([pd1da,pd1c],E=T+P);
```

$$E=\frac{h^2\,m\,(2\,e\cos(\theta-BB)+e^2+1)}{2\,p^2}-\frac{m\,\Gamma\,(e\cos(\theta-C)+1)}{p}$$

use:p6e3a

use : BB=C

```
pdf1:ratsimp(subst([p6e3a,BB=C],pdf));
```

$$E=\frac{(e^2-1)\,m\,\Gamma^2}{2\,h^2}$$

```
pdf2:combine(expand(pdf1));
```

$$E=\frac{e^2\,m\,\Gamma^2-m\,\Gamma^2}{2\,h^2}$$

```
pdf3:solve(pdf2,e);
```

$$\left[e=-\left(\frac{\sqrt{\Gamma^2+\frac{2\,E\,h^2}{m}}}{\Gamma}\right),e=\frac{\sqrt{\Gamma^2+\frac{2\,E\,h^2}{m}}}{\Gamma}\right]$$

use pdf3, because all values are pos

```
pdf4:pdf3[2];
```

$$e=\frac{\sqrt{\Gamma^2+\frac{2\,E\,h^2}{m}}}{\Gamma}$$

Orbits (= Trajectory of the small mass = m around a large mass =M)
==========

```
Orb: matrix(
[circle, e=0, Nan],
 [Ellipse,e<1,1/2·m·V^2<G·M·m/r],
 [Parabola,e=1,1/2·m·V^2=G·M·m/r],
 [Hyperbola,e>1,1/2·m·V^2>G·M·m/r]
);
```

$$\begin{pmatrix} circle & e=0 & Nan \\ Ellipse & e<1 & \frac{V^2 m}{2} < \frac{G M m}{r} \\ Parabola & e=1 & \frac{V^2 m}{2} = \frac{G M m}{r} \\ Hyperbola & e>1 & \frac{V^2 m}{2} > \frac{G M m}{r} \end{pmatrix}$$

```
pcf:E = 1/2·m·V^2−Γ·m/r;
```

$$E = \frac{V^2 m}{2} - \frac{m \Gamma}{r}$$

a) computation , e=0, circular velocity
---

use :pdf2

```
pcf1:subst([e=0],pdf2);/*circle,info from table*/
```

$$E = -\left(\frac{m \Gamma^2}{2 h^2}\right)$$

```
pcf2:expand(subst([pcf],pcf1)/m);
```

$$\frac{V^2}{2} - \frac{\Gamma}{r} = -\left(\frac{\Gamma^2}{2 h^2}\right)$$

use:pe3c1

```
pcf3:reverse(subst([e=0],pe3c1));
```

$$r^2 = p^2$$

use :p6e3a

```
pcf4:reverse(p6e3a·Γ);
```

$$h^2 = p \Gamma$$

subst : pcf4 into pcf2,use pcf3

```
pcf5:subst([pcf4,p=r],pcf2);
```

$$\frac{V^2}{2} - \frac{\Gamma}{r} = -\left(\frac{\Gamma}{2 r}\right)$$

```
pcf6:solve(pcf5,V);
```

$$\left[V = -\left(\frac{\sqrt{\Gamma}}{\sqrt{r}}\right), V = \frac{\sqrt{\Gamma}}{\sqrt{r}}\right]$$

because both pos : e=0

```
print( "V:Circular Velocity =", (rhs(pcf6[2])))$
```

$$V{:}Circular\ Velocity = \frac{\sqrt{\Gamma}}{\sqrt{r}}$$

b) computation , e=1, parabolic velocity

```
pbf:subst([E=0],pcf);
```

$$0 = \frac{V^2 m}{2} - \frac{m \Gamma}{r}$$

```
pbf1:solve(pbf,V);
```

$$\left[V = -\left(\frac{\sqrt{2}\sqrt{\Gamma}}{\sqrt{r}}\right), V = \frac{\sqrt{2}\sqrt{\Gamma}}{\sqrt{r}}\right]$$

```
print( "V:Parabolic Velocity =", (rhs(pbf1[2])))$
```

$$V{:}Parabolic\ Velocity = \frac{\sqrt{2}\sqrt{\Gamma}}{\sqrt{r}}$$

that is called escape velocity : r=re ,radius of earth.

example : computation , 0< e< 1,elliptic orbit
---

rocket launch (= space vehicle) burnout velocity t @ 8km/s above the earth , position of rocket 30 ° degree above earth equator at distance above earth 8000 km where rocket burnout ,velocity angle 10° above the horizontal horizon .
==========
rem : horizontal horizon is perpendicular to burnout distance (= RE +height above the eart)
==========
compute : the elements of the orbit :

```
r = p /(1+e*cos(θ-C))

solution :

1e) height above the earth  [m]
---------------------------------------------
r h=Re+hh
```

```
Re:6.44·10^6;/*radius of the earth*/
rh : 8.0 ·10^6;/*given*/
```

6440000.0 8000000.0

```
q1a:hh =rh−Re;
```

$hh = 1560000.0$

2e) Γ = G*Me

```
Γ:3.986·10^14;/*m^3/s^2*/
```

$3.986\ 10^{14}$

3e) h = r*V0 = r*V*cos(C)

```
V: 8000;/*m/s*/
C:float(7·π/180) ;/*rad*/
sa:h = rh·V·cos(C);
```

8000 0.12217304763960307 $h = 6.352295370504461\ 10^{10}$

4e) p=h^2/Γ

```
p:ev(subst([%],h^2/Γ),nouns);
```

$1.0123345828934371\ 10^{7}$

5e) e=sqrt(1+2*h^2*E/(m*Γ^2))
---

m : not given E→ Em,see above.

```
e:sqrt(1+2·h^2·Em/Γ^2);
```

$$\sqrt{1.258796152816846\ 10^{-29}\ Em\ h^2+1}$$

use :p1cf

```
q2:expand(pcf/m);
```

$$\frac{E}{m}=\frac{V^2}{2}-\frac{\Gamma}{r}$$

```
q2a:subst([E/m=Em,r=rh],q2);
```

$$Em=\frac{V^2}{2}-1.25\ 10^{-7}\ \Gamma$$

```
q2b:ev(q2a,nouns);
```

$$Em=-1.7825\ 10^{7}$$

find :e

```
e:subst([q2b, sa],e);
```

0.307551394904985

6e) " conic section"

```
q2c:pe3b;
```

$$r=\frac{p}{e\cos(\theta-BB)+1}$$

```
q2c1:ev(q2c,nouns);
```

$$r=\frac{1.0123345828934371\ 10^{7}}{0.307551394904985\cos(\theta-BB)+1}$$

a) How to find : BB

BB is the phase angle =? , r=rh & θ = 0 (@ burnout)

```
q2c2:subst([r=rh,θ=0],q2c1);
```

$$8000000.0=\frac{1.0123345828934371\ 10^{7}}{0.307551394904985\cos(BB)+1}$$

```
q2c3:solve(q2c2,BB);
```

solve: using arc-trig functions to get a solution.

Some solutions will be lost. $\left[BB=\operatorname{acos}\left(\frac{750250481621386347}{869347155820000000}\right)\right]$

```
q2c4:float(q2c3);
```

$[BB=0.529609391730455]$

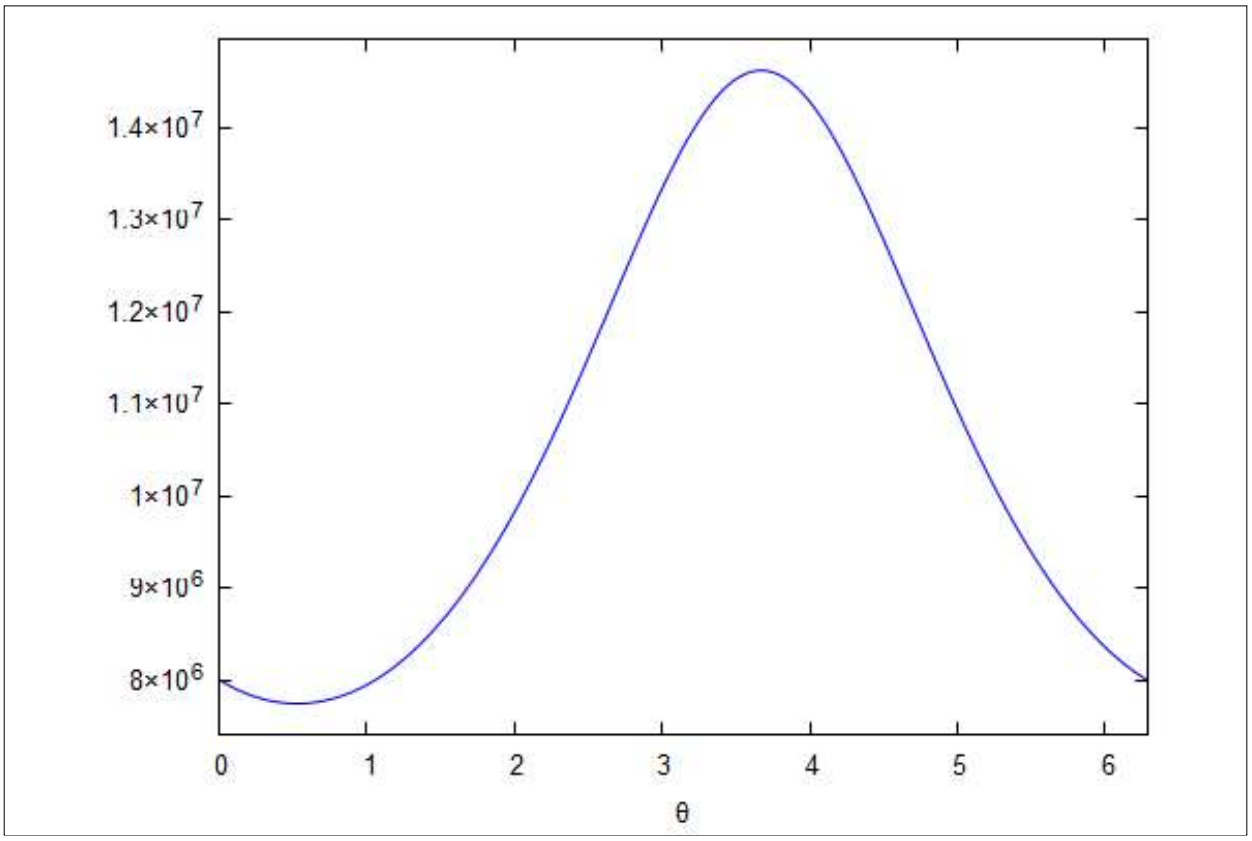
b) general equation : conic
---

use : q2c1

```
q2c5:subst(q2c4,q2c1);
```

$$r=\frac{1.0123345828934371\ 10^{7}}{0.307551394904985\ \cos(\theta-0.529609391730455)+1}$$

```
wxplot2d(rhs(q2c5), [θ,0,2·%pi])$
```



because : value negative ,because cos(-x) =cos(x),automatic simplification.
